FINITE DIFFERENCE SOLUTION OF LAPLACE'S EQUATION

I. INTRODUCTION

The potentials relationship to the charge distribution can be expressed either in differential or integral form

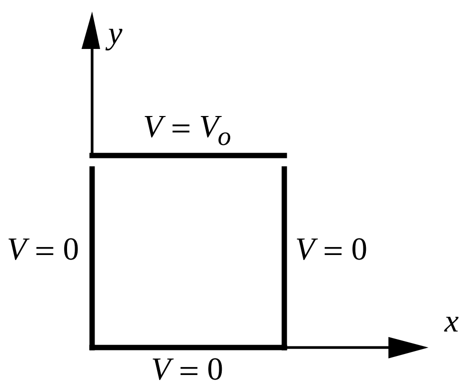
$$\nabla^2 V = -\rho/\varepsilon_o$$
$$V = \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{\rho}{\left|\vec{R}-\vec{R}'\right|}dv'$$

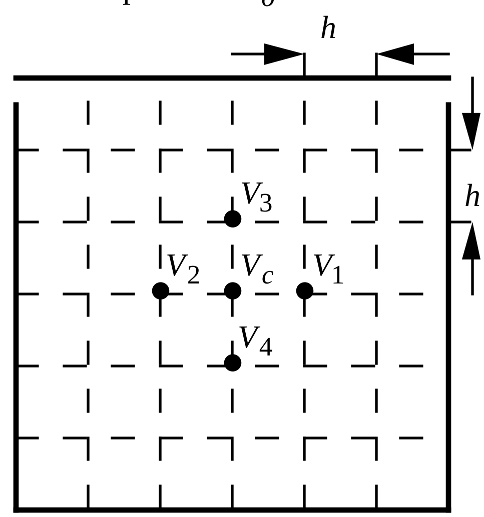
The differential form is usually solved by approximating the ∇ operator by finite differences. The integral equation is solved by approximating ρ by a series with unknown expansion coefficients, and then applying the boundary conditions to find the constants.

II. EXAMPLE: POTENTIAL IN A TROUGH

1. Finite Difference Formulation

Consider a two-dimensional problem such as an infinitely long trough with the cross section shown below. Three sides are grounded and the fourth is at a potential V_o.





For a two-dimensional geometry Laplace's equation becomes

$$\nabla^2 V(x,y) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V(x,y)$$

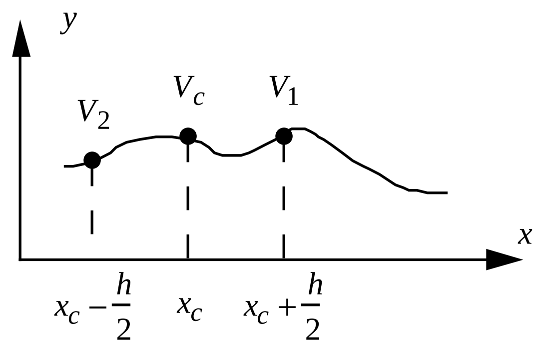
To apply the finite difference approximation the source free region inside of the trough is divided into cells, in this case shown as square. Each node inside is the center of a five-point cross of nodes with potentials $V_c, V_1, V_2, V_3,$ and V_4, Using the central difference method the first and second derivatives are represented by the following finite differences:

$$\left.\frac{dV}{dx}\right|_{x_c} \approx \frac{V_1 - V_2}{2h}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x_c} \approx \frac{\left.\frac{dV}{dx}\right|_{x_c+h/2} - \left.\frac{dV}{dx}\right|_{x_c-h/2}}{h}$$

$$\approx \frac{\frac{V_1 - V_c}{h} - \frac{V_c - V_2}{h}}{h}$$

$$\approx \frac{V_1 - 2V_c + V_2}{h^2}$$



Similarly for the y component

$$\frac{d^2V}{dy^2} \approx \frac{V_3 - 2V_c + V_4}{h^2}$$

and Laplace's equation becomes

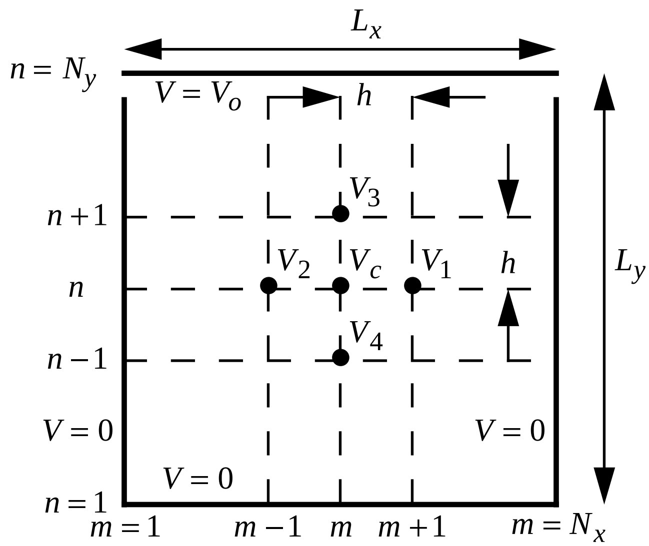
$$\nabla^2 V \approx \frac{V_1 + V_2 + V_3 + V_4 - 4V_c}{h^2} = 0$$

In order to find the potential everywhere inside of the trough, this equation must be applied at all of the interior nodes. For convenience the nodes are renumbered with index m along the x axis and n along the y axis. The number of cells is

$$N_x = \frac{L_x}{h} + 1 \qquad N_y = \frac{L_y}{h} + 1$$

and the total number of nodes inside of the trough is $N = (N_x - 2)(N_y - 2)$. The interior nodes run from $m = 2$ to $m = N_x$ along the x axis and from $n = 2$ to $n = N_y$ along the y axis. At node (m,n) Laplace's equation is

$$\frac{1}{h^2}\left[V(m+1,n) + V(m-1,n) + V(m,n+1) + V(m,n-1) - 4V(m,n)\right] = 0$$



2. Boundary Conditions

The boundary conditions are now imposed:

1. $V(m,n) = 0$	if	$m = 1$	and	$1 \le n \le N_y$	(left wall)
		$m = N_x$	and	$1 \le n \le N_y$	(right wall)
		$n = 1$	and	$1 \le m \le N_x$	(bottom)
2. $V(m,n) = V_o$	if	$n = N_y$	and	$1 \le m \le N_x$	(top)

Next, number the nodes sequentially using a single index

$$V_k \equiv V(m,n) \quad \text{where } k = (m - 1) + (N_x - 2)(n - 2) \quad \text{for } 1 \le k \le N$$

There are N equations as follows:

#1: $k = 1 (m = n = 2)$: $$\frac{1}{h^2}\left[V(3,2) + \underbrace{V(1,2)}_{=0} + V(2,3) + \underbrace{V(2,1)}_{=0} - 4V(2,2)\right] = 0$$

#2: $k = 2 (m = 3, n = 2)$: $$\frac{1}{h^2}\left[V(4,2) + V(2,2) + V(3,3) + V(3,1) - 4V(3,2)\right] = 0$$

$$\vdots$$

$\# N: \;\; k = N(m = N_x - 1, n = N_y - 1):$

$$\frac{1}{h^2}\left[\underbrace{V(N_x, N_y - 1)}_{=0} + V(N_x - 2, N_y - 1) + \underbrace{V(N_x - 1, N_y)}_{V_o} + V(N_x - 1, N_y - 2) - 4V(N_x - 1, N_y - 1)\right] = 0$$

The equations are rewritten with the excitation terms V_o on the right-hand sides. For example, equation number N,

$$\frac{1}{h^2}\left[V(N_x - 2, N_y - 1) + V(N_x - 1, N_y - 2) - 4V(N_x - 1, N_y - 1)\right] = -V_o$$

3. Matrix Equation

The N equations can be cast into matrix form:

Node k	V_1 (2,2)	V_2 (3,2)	V_3 (4,2)	…	V_{N_x-2} $(N_x - 1, 2)$	V_{N_x-1} (2,3)	V_{N_x} (3,3)	…	V_{N-1}	V_N
1	-4	1	0	…	0	1	0	…	0	0
2	1	-4	1	…	0	0	1	…	0	0
⋮	⋮	⋮	⋮	…	⋮	⋮	⋮	…	⋮	⋮
N	0	0	0	…	0	0	0	…	1	-4

The entries in the table are defined as the matrix **Q.** The boundary excitation vector is

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -V_o \\ -V_o \\ \vdots \\ -V_o \end{pmatrix}$$

and the vector of unknown voltages

$$\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix}$$

so that **QV**=**E.** Solve for the potentials by inverting **Q** and pre-multiplying

$$\mathbf{V} = \mathbf{Q}^{-1}\mathbf{E}$$

4. Sample Data

Result for a sample calculation follows: $L_x = 1$ m, $L_y = 2$ m, $h = 0.0625$ m, $N_x = 20$, $N_y = 40$, $V_o = 100\,\mathrm{V}$.